Birman-Wenzl-Murakami Algebra and the Topological Basis

Zhou Chengcheng,[1,*] Xue Kang,[1,†] Wang Gangcheng,[1] Sun Chunfang,[1] and Du Guijiao[1]

[1] *School of Physics, Northeast Normal University, Changchun 130024, People's Republic of China*

In this paper, we use entangled states to construct 9×9-matrix representations of Temperley-Lieb algebra (TLA), then a family of 9×9-matrix representations of Birman-Wenzl-Murakami algebra (BWMA) have been presented. Based on which, three topological basis states have been found. And we apply topological basis states to recast nine-dimensional BWMA into its three-dimensional counterpart. Finally, we find the topological basis states are spin singlet states in special case.

PACS numbers: 03.65.Ud,02.10.Kn,02.10.Yn

* Zhoucc237@nenu.edu.cn

† Xuekang@nenu.edu.cn

I. INTRODUCTION

Quantum entanglement (QE) is the most surprising nonclassical property of quantum systems which plays a key role in quantum information and quantum computation processing[1–4]. Because of these applications, QE has become one of the most fascinating topics in quantum information and quantum computation. To the best of our knowledge, the Yang-Baxter equation (YBE) plays an important role in quantum integrable problem, which was originated in solving the one-dimensional δ-interacting models[5] and the statistical models[6]. Braid group representations (BGRs) can be obtained from YBE by giving a particular spectral parameter. BGRs of two and three eigenvalues have direct relationship with Temperley-Lieb algebra (TLA)[7] and Birman-Wenzl-Murakami algebra (BWMA)[8] respectively. TLA and BWMA have been widely used to construct the solutions of YBE[9–12].

The TLA first appeared in statistical mechanics as a tool to analyze various interrelated lattice models[7] and was related to link and knot invariants[13]. In the subsequent developments TLA is related to knot theory, topological quantum field theory, statistical physics, quantum teleportation, entangle swapping and universal quantum computation[14, 15]. On the other hand, the BWMA[8] including braid algebra and TLA was first defined and independently studied by Birman, Wenzl and Murakami. It was designed partially help to understand Kauffman's polynomial in knot theory. Recently, Ref.[16] applied topological basis states for spin-1/2 system to recast 4-dimensional YBE into its 2-dimensional counterpart. As we know, few studies have reported topological basis states for spin-1 system. The motivation for our works is to find topological basis states for spin-1 system and study the topological basis states.

The purpose of this paper is twofold: one is that we construct a family of 9×9-matrix representations of BWMA; the other concerns topological basis states for spin-1 system. This paper is organized as follows. In Sec. 2, we use entangled states to construct the 9×9 matrix representations of TLA, then we present a family of 9×9-matrix representations of BWMA, and study the entangled states. In Sec. 3, we obtain three topological basis states of BWMA, and we recast nine-dimensional BWMA into its three-dimensional counterpart. We end with a summary.

II. 9×9-MATRIX REPRESENTATIONS OF BWMA

The 4×4 Hermitian matrix E, which satisfies TLA and can construct the well-known six-vertex model[17], takes the representation

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & q & \eta & 0 \\ 0 & \eta^{-1} & q^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1}$$

where $\eta = e^{i\varphi}$ with φ being any flux. We can rewrite E as that

$$\begin{cases} E = d|\Psi\rangle\langle\Psi|, \\ |\Psi\rangle = d^{-1/2}(q^{1/2}|\uparrow\downarrow\rangle + q^{-1/2}e^{-i\varphi}|\downarrow\uparrow\rangle), \end{cases} \tag{2}$$

where $d = q + q^{-1}$.

So like this symmetrical method, we found the 9×9 Hermitian matrices E's, which satisfies TLA, take the representations as follows

$$\begin{cases} E = d|\Psi\rangle\langle\Psi|, \\ |\Psi\rangle = d^{-1/2}(q^{1/2}|\lambda\mu\rangle + e^{i\phi_\nu}|\nu\nu\rangle + q^{-1/2}e^{i\varphi_{\mu\lambda}}|\mu\lambda\rangle), \end{cases} \tag{3}$$

where $d = q + 1 + q^{-1}$, $\lambda \neq \mu \neq \nu \in (1, 0, -1)$ and $(d, q, \phi_\nu, \varphi_{\mu\lambda}) \in real$. Recently, a 9×9–matrix representation of BWMA has been presented[18, 19]. We notice that E is the same as Gou *et.al.*[18, 19] presented, when $\phi_\nu = \varphi_2 - \varphi_1 + \pi$, $\varphi_{\mu\lambda} = -2\varphi_1$, $\lambda = 1$, $\mu = -1$ and $\nu = 0$.

As we know the BWMA relations[8, 9, 20, 21] including braid relations and TLA relations satisfy the following

relations,

$$\begin{cases} S_i - S_i^{-1} = \omega(I - E_i), \\ S_i S_{i\pm1} S_i = S_{i\pm1} S_i S_{i\pm1},\ S_i S_j = S_j S_i, |i-j| \geq 2, \\ E_i E_{i\pm1} E_i = E_i,\ E_i E_j = E_j E_i,\ |i-j| \geq 2, \\ E_i S_i = S_i E_i = \sigma E_i, \\ S_{i\pm1} S_i E_{i\pm1} = E_i S_{i\pm1} S_i = E_i E_{i\pm1}, \\ S_{i\pm1} E_i S_{i\pm1} = S_i^{-1} E_{i\pm1} S_i^{-1}, \\ E_{i\pm1} E_i S_{i\pm1} = E_{i\pm1} S_i^{-1},\ S_{i\pm1} E_i E_{i\pm1} = S_i^{-1} E_{i\pm1}, \\ E_i S_{i\pm1} E_i = \sigma^{-1} E_i, \\ E_i^2 = \left(1 - \dfrac{\sigma - \sigma^{-1}}{\omega}\right) E_i, \end{cases} \tag{4}$$

where $S_i, S_{i\pm1}$ satisfy the braid relations, $E_i, E_{i\pm1}$ satisfy the TLA relations [7]

$$\begin{cases} E_i E_{i\pm1} E_i = E_i,\ E_i E_j = E_j E_i,\ |i-j| \geq 2, \\ E_i^2 = d E_i, \end{cases} \tag{5}$$

where $0 \neq d \in \mathbb{C}$ is topological parameter in the knot theory which does not depend on the sites of lattices. We denote $\sigma = q^{-2}$ and $\omega = q - q^{-1}$ throughout the text. The notations $E_i \equiv E_{i,i+1}$ and $S_i \equiv S_{i,i+1}$ are used, $E_{i,i+1}$ and $S_{i,i+1}$ are abbreviation of $I_1 \otimes ... \otimes I_{i-1} \otimes E_{i,i+1} \otimes I_{i+2} \otimes ... \otimes I_N$ and $I_1 \otimes ... \otimes I_{i-1} \otimes S_{i,i+1} \otimes I_{i+2} \otimes ... \otimes I_N$ respectively, and I_j represents the unit matrix of the j-th particle.

Following the matrix representation of TLA we obtain a family of 9×9-matrix representations of BWMA as follows

$$\begin{cases} E = d|\Psi\rangle\langle\Psi|, \\ |\Psi\rangle = d^{-1/2}(q^{1/2}|\lambda\mu\rangle + e^{i\phi_\nu}|\nu\nu\rangle + q^{-1/2}e^{i\varphi_{\mu\lambda}}|\mu\lambda\rangle), \end{cases} \tag{6}$$

$$\begin{aligned} S = {} & q(|\lambda\lambda\rangle\langle\lambda\lambda| + |\mu\mu\rangle\langle\mu\mu|) + |\nu\nu\rangle\langle\nu\nu| \\ & + (q - q^{-1})(|\nu\lambda\rangle\langle\nu\lambda| + |\mu\nu\rangle\langle\mu\nu|) + (q-1)^2(q+1)q^{-2}|\mu\lambda\rangle\langle\mu\lambda| \\ & + e^{-i\varphi_{\mu\lambda}/2}(|\lambda\nu\rangle\langle\nu\lambda| + |\nu\mu\rangle\langle\mu\nu|) + e^{i\varphi_{\mu\lambda}/2}(|\nu\lambda\rangle\langle\lambda\nu| + |\mu\nu\rangle\langle\nu\mu|) \\ & + q^{-1}e^{-i\varphi_{\mu\lambda}}|\lambda\mu\rangle\langle\mu\lambda| + q^{-1}e^{i\varphi_{\mu\lambda}}|\mu\lambda\rangle\langle\lambda\mu| \\ & - q^{-3/2}(q^2-1)(e^{i(\phi_\nu - \varphi_{\mu\lambda})}|\nu\nu\rangle\langle\mu\lambda| + e^{-i(\phi_\nu - \varphi_{\mu\lambda})}|\mu\lambda\rangle\langle\nu\nu|), \end{aligned} \tag{7}$$

$$
\begin{aligned}
S^{-1} = & \, q^{-1}(|\lambda\lambda\rangle\langle\lambda\lambda| + |\mu\mu\rangle\langle\mu\mu|) + |\nu\nu\rangle\langle\nu\nu| \\
& + (q^{-1} - q)(|\lambda\nu\rangle\langle\lambda\nu| + |\nu\mu\rangle\langle\nu\mu|) + (q-1)^2(q+1)q^{-1}|\lambda\mu\rangle\langle\lambda\mu| \\
& + e^{-i\varphi_{\mu\lambda}/2}(|\lambda\nu\rangle\langle\nu\lambda| + |\nu\mu\rangle\langle\mu\nu|) + e^{i\varphi_{\mu\lambda}/2}(|\nu\lambda\rangle\langle\lambda\nu| + |\mu\nu\rangle\langle\nu\mu|) \\
& + qe^{-i\varphi_{\mu\lambda}}|\lambda\mu\rangle\langle\mu\lambda| + qe^{i\varphi_{\mu\lambda}}|\mu\lambda\rangle\langle\lambda\mu| \\
& + q^{-1/2}(q^2-1)(e^{-i\phi_\nu}|\lambda\mu\rangle\langle\nu\nu| + e^{i\phi_\nu}|\nu\nu\rangle\langle\lambda\mu|),
\end{aligned} \tag{8}
$$

where $d = q + 1 + q^{-1}$, $\lambda \neq \mu \neq \nu \in (1, 0, -1)$ and $(d, q, \phi_\nu, \varphi_{\mu\lambda}) \in real$.

It is worth noticing that the states $|\Psi\rangle$'s are entangled states. By means of negativity, we study these entangled states. The negativity for two qutrits is given by,

$$
\mathcal{N}(\rho) \equiv \frac{\| \rho^{T_A} \| - 1}{2}, \tag{9}
$$

where $\| \rho^{T_A} \|$ denotes the trace norm of ρ^{T_A}, which denotes the partial transpose of the bipartite state ρ [22]. In fact, $\mathcal{N}(\rho)$ corresponds to the absolute value of the sum of negative eigenvalues of ρ^{T_A} , and negativity vanishes for unentangled states[23]. By calculation, we can obtain the negativity of states $|\Psi\rangle$'s as

$$
\mathcal{N}(q) = \frac{q^{1/2} + 1 + q^{-1/2}}{d}, \tag{10}
$$

where $d = q+1+q^{-1}$. The Fig.1 corresponds to the negativity $\mathcal{N}(q)$. One demonstrates that the states $|\Psi\rangle$'s become maximally entangled states of two qutrits as $|\Psi\rangle = (|\lambda\mu\rangle + e^{i\phi_\nu}|\nu\nu\rangle + e^{i\varphi_{\mu\lambda}}|\mu\lambda\rangle)/\sqrt{3}$ when $q = 1$.

III. TOPOLOGICAL BASIS STATES

In the topological quantum computation theory, the two-dimensional (2D) braid behavior under the exchange of anyons[24] has been investigated based on the $\nu = 5/2$ fractional quantum Hall effect (FQHE)[25]. The orthogonal topological basis states read[25]

$$
\begin{aligned}
|e_1> &= \frac{1}{d}, \\
|e_2> &= \frac{1}{\sqrt{d^2-1}}(- \frac{1}{d}),
\end{aligned} \tag{11}
$$

where the parameter d represents the values of a unknotted loop. In Eq.(11) there are two topological graphics and . For four lattices, we can easy find four graphics , , , . If we use Skein relations $= q^{1/2} + q^{-1/2}$ $(S = q^{1/2}I + q^{-1/2}E)$ and $= q^{-1/2} + q^{1/2}$ $(S^{-1} = q^{-1/2}I + q^{1/2}E)$,

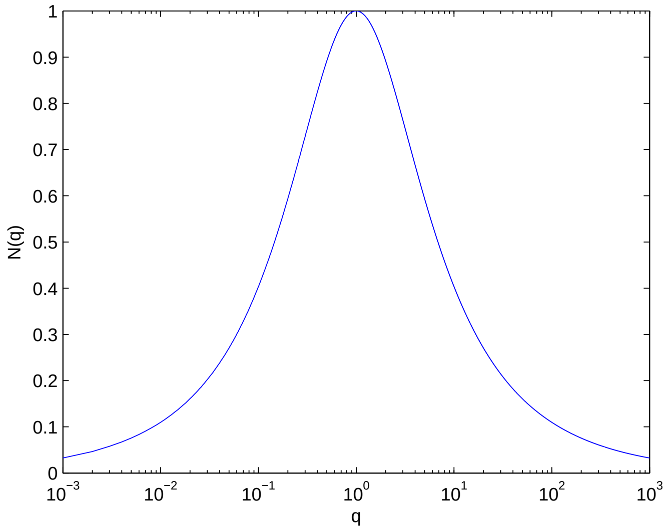


FIG. 1. (The negativity $\mathcal{N}(q)$ versus the parameter q.)

where the unknotted loop $d = \bigcirc = -q - q^{-1}$, the third and the fourth graphics recast to and ($= q^{1/2}$ $+ q^{-1/2}$, $= q^{-1/2}$ $+ q^{1/2}$). So the topological basis states(11) are self-consistent. But in this paper, we focus on BWMA, the braid group representations (S) is independent of TLA representations (E), and in BWMA $S - S^{-1} = \omega(I - E)$. So we know the graphics and have one independent graphic. We choose three independent graphics as , and .

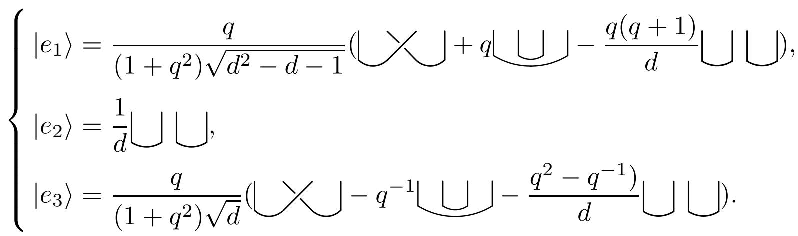
We define

$$
\begin{aligned}
&{}_{i}\ {}_{j} = d^{1/2}|\Phi_{ij}\rangle = q^{1/2}|\lambda\mu\rangle + e^{i\phi_\nu}|\nu\nu\rangle + q^{-1/2}|\mu\lambda\rangle,\\
&{}_{i}\ {}_{j} = d^{1/2}\langle\Phi_{ij}| = q^{1/2}\langle\lambda\mu| + e^{-i\phi_\nu}\langle\nu\nu| + q^{-1/2}\langle\mu\lambda|,\\
&[\ \ ]^\dagger = \ ,\\
&[\ \ ]^\dagger = \ ,\\
&[\ \ ]^\dagger = \ .
\end{aligned}
\tag{12}
$$

So E recasts to $E_{ij} =$. Following the BWMA, we define the graphic rules

$$
\begin{aligned}
&= \sigma\ ,\ = \sigma^{-1}\\
&- \ = \omega(\ -\ ),\\
&\bigcirc = d\ (\textit{the unknotted loop}).
\end{aligned}
\tag{13}
$$

The orthogonal basis states read

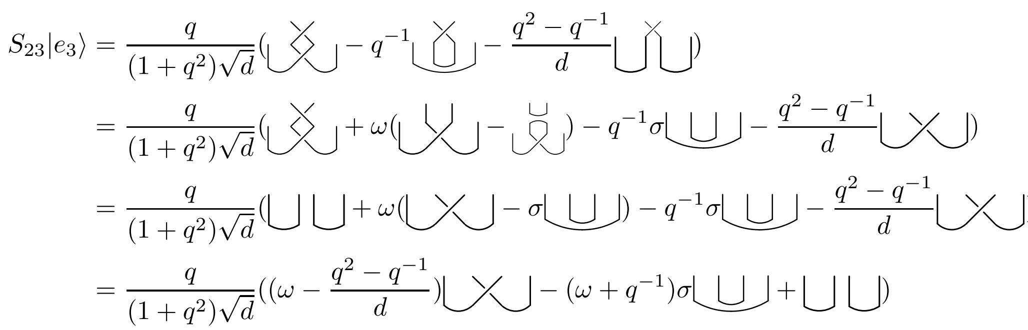


(14)

Let's introduce the reduced operators E_A, E_B, A and B

$$\begin{cases} (E_A)_{ij} = \langle e_i|E_{12}|e_j\rangle, \\ (E_B)_{ij} = \langle e_i|E_{23}|e_j\rangle, \\ A_{ij} = \langle e_i|S_{12}|e_j\rangle, \\ B_{ij} = \langle e_i|S_{23}|e_j\rangle. \end{cases} \tag{15}$$

Due to the limited length, we only show how S_{23} acts on $|e_3\rangle$ in detail as follows



$$= -\frac{\sqrt{d^2-d-1}}{q^2\sqrt{d}(d-1)}|e_1\rangle + \frac{q}{\sqrt{d}}|e_2\rangle + \frac{d-2}{d-1}|e_3\rangle. \tag{16}$$

Thus their matrix representations in the basis states $(|e_1\rangle, |e_2\rangle, |e_3\rangle)$ are given by

$$E_A = diag\{0, d, 0\}, \tag{17}$$

$$E_B = \begin{pmatrix} \frac{d^2-d-1}{d} & \frac{\sqrt{d^2-d-1}}{d} & -\frac{\sqrt{d^2-d-1}}{\sqrt{d}} \\ \frac{\sqrt{d^2-d-1}}{d} & \frac{1}{d} & -\frac{1}{\sqrt{d}} \\ -\frac{\sqrt{d^2-d-1}}{\sqrt{d}} & -\frac{1}{\sqrt{d}} & 1 \end{pmatrix}, \tag{18}$$

$$A = diag\{q, q^{-2}, -q^{-1}\}, \tag{19}$$

$$B=\begin{pmatrix}\frac{1}{q^4(d-1)d} & \frac{\sqrt{d^2-d-1}}{dq} & -\frac{\sqrt{d^2-d-1}}{q^2(d-1)\sqrt{d}}\\ \frac{\sqrt{d^2-d-1}}{dq} & \frac{q^2}{d} & \frac{q}{\sqrt{d}}\\ -\frac{\sqrt{d^2-d-1}}{q^2(d-1)\sqrt{d}} & \frac{q}{\sqrt{d}} & \frac{d-2}{d-1}\end{pmatrix}, \tag{20}$$

where E_A, E_B, A and B are Hermitian matrices. It is worth noting that $E_B = UE_AU^{-1}, B = UAU^{-1}$,

$$U=\begin{pmatrix}\frac{1}{(d-1)d} & -\frac{\sqrt{d^2-d-1}}{d} & -\frac{\sqrt{d^2-d-1}}{\sqrt{d}(d-1)}\\ \frac{\sqrt{d^2-d-1}}{d} & -\frac{1}{d} & \frac{1}{\sqrt{d}}\\ \frac{\sqrt{d^2-d-1}}{\sqrt{d}(d-1)} & \frac{1}{\sqrt{d}} & -\frac{d-2}{d-1}\end{pmatrix}, \tag{21}$$

and they satisfy the reduced BWMA relations

$$\begin{cases} A-A^{-1}=\omega(I-E_A),\ B-B^{-1}=\omega(I-E_B),\\ ABA=BAB,\\ E_AE_BE_A=E_A,\ E_BE_AE_B=E_B,\\ E_AA=AE_A=\sigma E_A,\ E_BB=BE_B=\sigma E_B,\\ ABE_A=E_BAB=E_BE_A,\ BAE_B=E_ABA=E_AE_B,\\ AE_BA=B^{-1}E_AB^{-1},\ BE_AB=A^{-1}E_BA^{-1},\\ E_AE_BA=E_AB^{-1},\ E_BE_AB=E_BA^{-1},\\ AE_BE_A=B^{-1}E_A,\ BE_AE_B=A^{-1}E_B,\\ E_ABE_A=\sigma^{-1}E_A,\ E_BAE_B=\sigma^{-1}E_B,\\ E_A^2=(1-\frac{\sigma-\sigma^{-1}}{\omega})E_A,\ E_B^2=(1-\frac{\sigma-\sigma^{-1}}{\omega})E_B. \end{cases} \tag{22}$$

We emphasize that (22) acts on the basis $(|e_1\rangle, |e_2\rangle, |e_3\rangle)$.

It is worth noting that the topological basis states are singlet states, when $\phi_\nu=\pi$, $\lambda=1$, $\mu=-1$, $\nu=0$ and $q=1$. In other words, $S^2|e_i\rangle=0$ and $S_z|e_i\rangle=0$, where $S=\sum_1^4 S_j$, S_j are the operators of spin-1 angular momentum for the j-th particle, $i=1,2,3$.

IV. SUMMARY

In this paper we construct 9×9-matrix representations of TLA, where we used the entangled states ($|\Psi\rangle = d^{-1/2}(q^{1/2}|\lambda\mu\rangle + e^{i\phi_\nu}|\nu\nu\rangle + q^{-1/2}e^{i\varphi_{\mu\lambda}}|\mu\lambda\rangle)$). Then we get a family of 9×9 representations of BWMA. We study

the entangled states $|\Psi\rangle$'s, and find the negativity related parameter q. The negativity became the maximum value if $q = 1$. In Sce. 3, we defined the third topological graphic  and find three orthogonal topological basis states of BWMA, based on the former researchers. It was mentioned that the Hermitian matrices E_A, E_B, A and B have an interesting similar transformation matrix U which satisfies $B = UAU^{-1}$ and $E_B = UE_AU^{-1}$. Based on them, we obtain a three-dimensional representation of BWMA. Finally we find the topological basis states are the spin singlet states, if $\phi_\nu = \pi$, $\lambda = 1$, $\mu = -1$, $\nu = 0$ and $q = 1$. Our next work will study how the topological basis states play a role in quantum theory.

V. ACKNOWLEDGMENTS

This work was supported by NSF of China (Grant No.10875026)

[1] C.H. Bennett and D.P. DiVincenzo, *Nature* **404** (2003) 247.

[2] C.H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W.K. Wootters, *Phys. Rev. Lett.* **70** (1993) 1895.

[3] C.H. Bennett, and S.J. Wiesner, *Phys. Rev. Lett.* **69** (1992) 2881.

[4] M. Murao, D. Jonathan, M.B. Plenio, and V. Vedral, *Phys. Rev. A* **59** (1999) 156.

[5] C.N. Yang, *Phys. Rev. Lett.* **19** (1967) 1312; C.N. Yang, *Phys. Rev.* **168** (1968) 1920.

[6] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (New York: Academic) (1982);
R.J. Baxter, *Ann. Phys.* **70** (1972) 193.

[7] H.N.V. Temperley and E.H. Lieb, *Proc. Roy. Soc. London,* **A 322** (1971) 25.

[8] J. Birman, H. Wenzl, *Trans. A.M.S.* **313** (1989) 249;
J. Murakami, *Osaka J. Math.* **24** (1987) 745.

[9] Y. Cheng, M.L. Ge, K. Xue, *Commun Math Phys.* **136** (1991) 195.

[10] V. Jones, *Commun. Math. Phys.* **125** (1989) 459.

[11] M.L. Ge, Y.S. Wu and K. Xue, *Inter. J. Mod. Phys.* **A6** (1991) 3735.

[12] G.C. Wang, K. Xue, *et al.*, *Quantum Information Processing* **9(6)** (2009) 699.

[13] M. Wadati, T. Deguchi, and Y. Akutsu, *Phys. Rep.* **180** (1989) 247332.

[14] Y. Zhang, *J. Phys. A: Math. Gen.* **39** (2006) 11599.

[15] L.H. Kauffman, *Knots and Physics* (Singapore: World Scientific Publ CoLtd.) (1991)

[16] S.W. Hu, K.xue, and M.L. Ge, *Phys. Rev. A* **78** (2008) 022319.

[17] M. Jimbo, *Yang-Baxter Equations on Integrable Systems* (World Scientific, Singapore) (1990)

[18] L.D. Gou, *et al.*, *Commun. Theor. Phys.* **55** (2011) 263.

[19] L.D. Gou, *et al.*, *International Journal of Quantum Information* **8** (2010) 1187.

[20] M.L. Ge and K. Xue, *J. Phys. A: Math.* **26** (1993) 281.

[21] V.F.R. Jones, *Commun Math Phys.* **125** (1987) 459.

[22] K. Zyczkowski, P. Horodecki, A. sanpera, M. lewenstein, *Phys. Rev. A.* **58** (1998) 883.

[23] G. Vidal, R.F. Werner, *Phys. Rev. A* **65** (2001) 032314.

[24] F. Wilczek, *Phys. Rev. Lett.* **48** (1982) 1144; G. Moore and N. Read, *Nucl. Phys. B* **360** (1991) 362.

[25] E. Ardonne, and K. Schoutens, *Ann. Phys.* **322** (2007) 21; A. Feiguin, S. Trebst, A.W.W. Ludwig, M. Troyer, A. Kitaev, Z.H. Wang, and M.H. Freedman, *Phys. Rev. Lett.* **98** (2007) 160409. for a review, see, C. Nayak, S.H. Simon, A. Stern, M. Freedman, and S.D. Sarma, *e-print arXiv:0707.1889*; K. Hikami, *Ann. Phys.* **323** (2008) 1729.